Archimedes and Approximations of π

Major Source: Journey Through Genius, William Dunham, chapter 4 (1990). A highly recommended book.

Suggestion: Don't read this without graph paper and pencil handy!

An ancient result is that the ratio of the circumference to the diameter is a constant. This apparently wasn't known as "pi" until post-Renaissance Europe.

Euclid proved that the ratio of the area of a circle to the square of its diameter was also a constant (possibly different from the above).

Archimedes proved that the formula for the area of a **regular** polygon is ½ h Q; where h is the height of the triangles formed by connecting the center of the polygon with the vertices (called the **apothem**), and Q is the perimeter of the polygon.

Euclid had shown that you could always inscribe a square inside a circle. If you bisect the sides of the square and construct perpendicular bisectors that intersect the circle – and use THOSE points – then you can inscribe a regular octagon. And so on ….

In fact, you can continue this process indefinitely to get as close to filling the circle as you want. In other words – if you want to get "arbitrarily close" to the area of a circle, there are regular inscribed polygons where (the Area of the circle) – (the Area of the regular inscribed polygon) < any arbitrarily small number. You can get as close as you need to.

This is called the "Method of Exhaustion" (due to Eudoxus or Archimedes) and is a wonderful pre-cursor to our modern intuitive idea of a "limit". In fact, the argument above suggests the epsilon-delta definition of a limit that some of you will read about (2.4 in Anton).

Archimedes then proved the following proposition:

Proposition: The area of a circle (called A) = the area of a right triangle with base = circumference of the circle, and the height = radius of the circle (this area is called T).

Proof:

Suppose that A > T.

We can construct an inscribed regular polygon (with area P) such that A – P < A – T.
And so, **T < P**.

However, P = ½ h Q, and h is certainly smaller than the radius, and Q is certainly smaller than the circumference, so P = ½ h Q < ½ r C = T. In other words, **P < T**.

This is a contradiction (P and T can't be simultaneously less than each other) and so A is **not** greater than T.

Next suppose that T < A. By an argument that is almost exactly the same (only considering circumscribed regular polygons) we get another contradiction. And so the proposition is proved.

This incidentally shows that the constant ratio of circumference to diameter, and the constant ratio of area to diameter (squared) are "strongly" related. (I made that terminology up …)

Next Archimedes used the circumferences of inscribed and circumscribed polygons of an increasing number of sides to narrow an estimate for pi.

Extra Credit:

You can follow the same process that Archimedes did to get your own lower or upper bounds for pi. He started with an inscribed and circumscribed hexagon because the arithmetic was easy – and you should too. Try to do as much as you can without calculators (except to check your work). Extra credit is available if you go to regular polygons with 24 sides. I'd suggest working with someone else, and have one of you do the inscribed polygons, and your partner should do the circumscribed polygons. The thinking will be easier I think …..

AND write it up nicely for me ☺

Other sources:

http://mathworld.wolfram.com/ArchimedesAlgorithm.html

http://mathworld.wolfram.com/ArchimedesRecurrenceFormula.html

Dörrie, H. *100 Great Problems of Elementary Mathematics: Their History and Solutions.* New York: Dover, p. 186, 1965.

And let me know if you find other good ones.